**Formal Properties of Green’s Functions**

Here are the 6 common green’s functions. For the Heisenberg picture operators, I think we must use the typical U(t,t0) = U0(t,t0)S(t,t0), as I don’t think the mixed phase convention time-development operator is compatible with the adiabatic theorem, which will be used later (see QM/Time-Dependent/Adiabatic Perturbation for discussion of phase convention). Note that the t argument is always with the A, and t′ always with the B. And the lesser/advanced GF are the only ones with the sign change.



These GF’s also implicitly carry with them exponential convergence factors that come in with the θ functions: θ(t) → θ(t)e-ηt, θ(-t) → θ(-t)eηt, where η = 0+ is an infinitesimal positive number, eventually taken to zero. Also, I think even the >,< GF’s carry these factors, and should be technically defined as, for instance,

G>(t,t´) = -iθ(t-t´)<AH(t)BH(t´)> - iθ(t´-t)<AH(t)BH(t´)>

→

G>(t,t´) = -iθ(t-t´)e-η(t-t´)<AH(t)BH(t´)> - iθ(t´-t)eη(t-t´)<AH(t)BH(t´)>.

Anyway, since this limit gives us 1 (unless t = ∞, or t´ = -∞), we will just leave it off, to save space, unless necessary. The exponential convergence factors are there, I think, because of the use of the adiabatic theorem:|GS> = Sη(t0,-∞)|GS0>, where the η is there to indicate exponentially slowly turning on the perturbation. This is where eηt factors would show up in calculations. And I believe this convergence factor, initially in S (see Quantum Mechanics/Time-Development/Adiabatic Perturbation), gets transferred to the GF’s and so effectively becomes part of their definition for any calculation involving times in infinite past or future (where there is a e-ηt term in the adiabatic theorem). For finite times, the exponential factors are superfluous, as their limit will always gives us 1.

In order for the Wick expansion, that we’ll use later, to hold, we must have that the background expectation state be an eigenstate of some bilinear H0 (or combination of such) |Ω0>



|Ω0> must also be ‘homogeneous’ – it cannot be a macroscopic quantum state, like a Bose condensate, which would seem to mean that the quantum occupation numbers must all be ~ 1/N. Furthermore this is only valid in the large V limit I think. The excitation vacuum |0> satisfies these restrictions. The operators AH(t), BH(t) can be anything, and may evolve in time according to some Hamiltonian,



where both the single particle and two particle interaction may be time-dependent, though the latter is never (?) realized. We will refer to both potentials, collectively, as simply V(t).

**Algebraic relationships between the various Green’s functions**

The green’s functions aren’t independent. Some algebraic relations between the green’s functions are as follows…well they’re just as they were in the single particle file:



**Symmetries of GF’s**

Consider a GF evaluated against some state |Ω0>. And presume time-development against a constant H. Constant H would mean energy is conserved. Then for instance our GF should depend only on t-t´, i.e., be time-translationally invariant.



Now if |Ω0> is an eigenstate of H, then:



But note that if we evaluated against states that weren’t eigenstates of H (like ones that were eigenstates of H0, rather), then this would not be time-translationally invariant.

**Green’s functions and the spectral function**

The following discussion will assume that we’re using the expectation basis, |Ω0>, which is an arbitrary state. We assume the operators develop according to a **time-independent** H for the spectral properties to hold. Now let’s look at the Fourier transform of the various green’s functions. These results also follow from the single particle file. I’ll just list them:



where,



(and Im = |m><m|) Note ω is real throughout. Let’s pause to observe that the poles of G are the *many-body excitations*. But they are modulated by the presence of Ω0. So only excitations that have non-zero overlap with |Ω0> will be present.

**Properties of the Spectral Function**

We can see that the spectral function can be inferred by:



If A(ω) is real, then GA = (GR)\*, and we can say:



Moreover, we saw the Spectral Function was explicitly given by:



So it clearly has poles (infinities) at the excitations of the system (since |m>, |n> are the eigenstates, and so En – Em would be the excitations). But these are modulated by the overlap of the excitation states with |Ω0> and the operators A and B. The reality of A(x) seems to hinge on a lot of things. So if A and B are Hermitian conjugates of each other (or themselves), and if |Ω0> is an eigenstate of H, then A(x) should be real since:



And then its complex conjugate would be:



So there.

**Kramers - Kronig relations obeyed by GF’s and A(ω)**

Repeating the arguments from the single particle file, we have, *presuming* A(ω) is real:



where the Hilbert transform is:



and for the spectral function, we have:



and concommitantly:



**Expectations from Green’s functions**

Consider a non-harmonic lattice Hamiltonian, in an external time-dependent field:



Suppose we are in some initial state |Ω0>. We let it evolve according to H, and then want to calculate the average spring potential energy as a function of time thereafter. Then we’d calculate:



We can extract this from a causal G, for instance. Define:



And then we’d have:



We could get KE too, using vH(R,t) = H(R,t),



Taking advantage of the differential equation for GC(t,t’), we might be able to get <xα(R)> and/or the total energy too, but whatever.

**Expectations from the spectral function**

From the definition of G<, we have:



We also have,



This property actually follows from the definition of A in terms of GR and GA. So it is completely general. Let’s consider calculating <AB>. We can clearly find the appropriate linear combination of the two above.



**Differential equations obeyed by GF’s**

Let’s consider an interacting phonon system.



where V is some function of the coordinates and time – could be cubic interaction, quartic, whatever.

**Retarded GF**

And let’s consider the retarded GF:



We can take the derivative w/r to time, and take advantage of the fact that the operator will obey classical equations of motion, to arrive at:



The commutator between the two x’s is zero. And so we’re left with derivative of x. So let’s work that out,



which is what we expect. So now we have:



Now let’s take the derivative again,



The first term can be evaluated most easily by factoring out the time-evolution of the operators, evaluating the commuator, and then applying the time-evolution operator again. Of course since the commutator is a constant, this cancels out. And we get:



and now we need to evaluate the derivative of α…



Assuming K is symmetric, we may write:



Filling this into our GF equation,



and so we have:



where the interacting force is:



**Lesser GF**

Consider the lesser GF. Then we have:



Taking time-derivative



We just found that:



So plugging that in, now we have:



Now let’s take the derivative again,



and now we need to evaluate the derivative of Pα…we already got this of course to be:



Filling this into our GF equation,



and so we have:



where the interacting force is:



I would imagine the greater, causal, advanced, etc., guys obey similar equations, but note that F>,C,A, etc., would also have corresponding changes in form. This is exactly the generalization of the single particle case that we’d expect. And note we’re not restricted to harmonic oscillator type problems, though these are the ones most amenable to perturbation theory I suppose.

**Index-dependence of GF**

Don’t know where else to put this. But if we have a homogeneous, isotropic, medium, then Gαβ should be diagonal in its indices. This is buttressed by the fact that Kαβ must be diagonal ~ Kδαβ (see that Navier Stokes file) as this is the only rank two tensor that’s homogeneous/isotropic, and this would necessarily make G(0)αβ diagonal. There is no homogeneous isotropic vector, so we couldn’t have single particle potentials, naturally. As there is no rank three tensor which is homogeneous/isotropic, so this would eliminate any cubic interaction. There are rank four tensors (see Navier Stokes again) that are homogeneous/isotropic and so a cuartic interaction could have non-trivial index dependence. But even still, we must presume that the GF would remain diagonal in its indices, as it is (?) a rank two tensor of sorts.